The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Homework 9
Please submit your assignment online on Blackboard
Due at 18:00 p.m. on Monday, 7th April, 2025

1. Suppose we have a box, and N balls in it. Initially, some of these balls are black and the rests are white. Now we repeatedly apply the following procedure:

   - Randomly choose one of the N balls with equal probability and take it out.
   - If the chosen ball is black, we put a white ball into the box.
     If the chosen ball is white, we put a black ball into the box.

   Let X_n be the number of black balls in the box after repeating the above procedure for independently n times. So we know $X = (X_n)_{n\geq 0}$ is a Markov chain with state space $S = \{0, 1, \cdots, N\}$ and the transition matrix P, which is given by

$$P(x,y) = \begin{cases} 1 - \frac{x}{N}, & y = x+1 \\ \frac{x}{N}, & y = x-1 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Prove that the Markov chain X is irreducible.
   By the theorem proved in class, there exists a stationary distribution

$$\mu = (\mu(0), \mu(1), \cdots, \mu(N)).$$

   (b) Recall that the stationary distribution μ satisfies $\mu^\top P = \mu^\top$, we obtain $N+1$ linear equations $\mu(n) = \sum_{k=0}^{N} \mu(k)P(k,n)$, for $n = 0, 1, \cdots, N$.
   Please simplify these equations for the transition matrix defined by (1).
   (For example, for $n = 0$, the linear equation is written as $\mu(1)/N = \mu(0)$.)

   (c) Prove that $\mu(2) = \frac{N(N-1)}{2}\mu(0)$ and $\mu(x) = \binom{N}{x}\mu(0)$ for all $x \in S$.

   (d) Compute the stationary distribution μ.

   Solution

   (a) Let x, y be two different states in S with $x < y$, then we have

$$P^{y-x}(x,y) = P(x,x+1)P(x+1,x+2)\cdots P(y-1,y) = (1 - \frac{x}{N})\cdots(1 - \frac{y-1}{N}) > 0$$

   and

$$P^{y-x}(y,x) = P(y,y-1)P(y-1,y-2)\cdots P(x+1,x) = \frac{y}{N}\cdots\frac{x+1}{N} > 0$$

   It follows that $x \leftrightarrow y$ and this Markov chain is irreducible.
   Since the chain is finite and irreducible, the stationary distribution exists and is unique.

(b) Note that $P(1,0) = \frac{1}{N}$ and $P(k,0) = 0$, $k \neq 1$, then

$$u(0) = \sum_{k=0}^{N} \mu(k)P(k,0) = \mu(1)P(1,0) = \mu(1) \cdot \frac{1}{N}.$$

For $n = 1, \cdots, N$, we applying $P(x, x+1) = 1 - \frac{x}{N}$ and $P(x, x-1) = \frac{x}{N}$ to derive

$$\begin{aligned}\mu(n) = \sum_{k=0}^{N} \mu(k)P(k,n) &= \mu(n-1)P(n-1,n) + \mu(n+1)P(n+1,n) \\ &= \mu(n-1) \cdot (1 - \frac{n-1}{N}) + \mu(n+1) \cdot \frac{n+1}{N}.\end{aligned}$$

(c) By part (b), we have $\mu(1) = N\mu(0)$ and

$$\mu(1) = \mu(0) + \mu(2)\frac{2}{N},$$

it follows that

$$\mu(2) = \frac{N}{2}(\mu(1) - \mu(0)) = \frac{N(N-1)}{2}\mu(0).$$

We use Mathematical Induction to prove $\mu(n) = \binom{N}{n}\mu(0)$ for all $n = 1, 2, 3, \cdots N$. First, $\mu(1) = N\mu(0) = \binom{N}{1}\mu(0)$ holds. Assume that $\mu(k) = \binom{N}{k}\mu(0)$ holds, then we prove $\mu(k+1) = \binom{N}{k+1}\mu(0)$. Since

$$\begin{aligned}u(k+1) &= \frac{N}{k+1}\mu(k) - \frac{N-k+1}{k+1}\mu(k-1) \\ &= \frac{N}{k+1}\binom{N}{k}\mu(0) - \frac{N-k+1}{k+1}\binom{N}{k-1}\mu(0) \\ &= \binom{N}{k+1}\mu(0).\end{aligned}$$

Hence $\mu(x) = \binom{N}{x}\mu(0)$ for all $x \in S$.

(d) Since μ is a stationary distribution, then $\mu(0) + \mu(1) + \cdots + \mu(N) = 1$ By substituting $\mu(x) = \binom{N}{x}\mu(0)$ into

$$\sum_{x=0}^{N} \mu(x) = \sum_{x=0}^{N} \binom{N}{x}\mu(0) = 1$$

yields

$$\mu(0) = \frac{1}{\sum_{x=0}^{N} \binom{N}{x}} = \frac{1}{2^N}.$$

It follows that

$$\mu(x) = \binom{N}{x}\frac{1}{2^N}.$$

2. Consider the simple random walk $X = (X_n)_{n \geq 0}$ with state space $\mathbb{Z}$ (the set of all integers) and transition matrix P, which is given by

$$P(i,j) = \begin{cases} 1/2, & j = i+1 \text{ or } j = i-1 \\ 0, & \text{otherwise.} \end{cases}$$

If π is a stationary distribution of X, then

(a) Prove that $\frac{\pi(x-1)+\pi(x+1)}{2} = \pi(x)$ for all $x \in \mathbb{Z}$.

(b) Let $u(x) = \pi(x) - \pi(x-1)$ for $x \in \mathbb{Z}$ and prove that $u(x) = C$ for some constant C for any $x \in \mathbb{Z}$.

(c) Prove that $\pi(x) = ax + b$ for some constant a, b.

(d) Prove that X does not have a stationary distribution.

Solution

(a) If π is a stationary distribution of X, then for all $x \in \mathbb{Z}$

$$\begin{aligned}\pi(x) &= \sum_{k\in\mathbb{Z}} \pi(k)P(k,x) = \pi(x-1)P(x-1,x) + \pi(x+1)P(x+1,x)\\ &= \frac{1}{2}\pi(x-1) + \frac{1}{2}\pi(x+1).\end{aligned}$$

(b) We rewrite $\frac{\pi(x-1)+\pi(x+1)}{2} = \pi(x)$ as

$$\frac{\pi(x+1)-\pi(x)}{2} = \frac{\pi(x)-\pi(x-1)}{2}$$

which implies $u(x+1) = u(x)$ holds for all $x \in \mathbb{Z}$. Hence $u(x) = C$ for some non zero constant C for any $x \in \mathbb{Z}$.

(c) Let x be an positive integer,using $u(x) = \pi(x) - \pi(x-1) = C$, we write

$$\pi(x) = C + \pi(x-1) = 2C + \pi(x-2) = \cdots = xC + \pi(0).$$

If x is an negative integer, using $u(x) = \pi(x+1) - \pi(x) = C$, we obtain

$$\pi(x) = \pi(0) - xC.$$

Therefore $\pi(x) = Cx + \pi(0)$ holds for $x \in \mathbb{Z}$.

(d) If π is a stationary distribution, then $\pi(x) = ax + b \geq 0$ and

$$\sum_{x\in\mathbb{Z}} \pi(x) = \sum_{x\in\mathbb{Z}} (ax+b) = 1$$

which is impossible. Hence we conclude that no stationary distribution exists.

3. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with state space $\mathbb{N}$ (the set of all nonnegative integers) and transition matrix P, which is given by

$$P(j,k) = \begin{cases} 1, & k = j-1,\ j \geq 1,\\ 0, & k \neq j-1,\ j \geq 1,\\ \nu(k), & k \in \mathbb{N},\ j = 0.\end{cases}$$

where $\nu = \{\nu(n)\}_{n\geq 0}$ is a probability measure on $\mathbb{N}$, i.e. $\nu(n) \geq 0$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \nu(n) = 1$.

(a) Prove that X is irreducible if and only if $\nu(\{n, n+1, \cdots\}) > 0$ for any $n \in \mathbb{N}$, where $\nu(\{n, n+1, \cdots\}) = \sum_{k=n}^{\infty} \nu(k)$.

(b) Prove that 0 is recurrent.

(c) Prove that the measure defined by $\mu(n) = \nu(\{n, n+1, \cdots\}), n \in \mathbb{N}$ is stationary, i.e. $\mu^\top P = \mu^\top$.

Solution

(a) If $\nu(\{n, n+1, \cdots\}) = \sum_{k=n}^{\infty} \nu(k) > 0$ holds for all $n \geq 0$, then

$$\lim_{n\to\infty} v(n) > 0$$

which implies that for sufficient large N, $0 \to N$. Obviously, $N \to 0$.

Next, we show that any two states in this chain is intercommunicate. Without loss generality, we assume i, j are two different states with $i < j$, then $P^{j-i}(j, i) = P(X_{j-i} = i \,|X_0 = j) = 1$, so $j \to i$. On the other hand, we can reach j from i by taking $i \to 0 \to N \to j$. There for state i and j are intercommunicate, and the chain is irreducible.

If the chain is irreducible, then all state are intercommunicate and $\nu(n) > 0$ for all $n \geq 0$. Hence $\nu(\{n, n+1, \cdots\}) > 0$ for any $n \in \mathbb{N}$.

(b) Since

$$P_0(\tau_0 = 1) = P(X_1 = 0|X_0 = 0) = v(0)$$

$$P_0(\tau_0 = 2) = P(X_2 = 0, X_1 = 1|X_0 = 0) = v(1)$$

and generally

$$P_0(\tau_0 = n) = P(X_n = 0, X_1 = 1, \cdots X_2 = n-2, X_1 = n-1|X_0 = 0) = v(n-1)$$

Hence

$$P_0(\tau_0 < \infty) = P(\tau_0 < \infty|X_0 = 0) = \sum_{k=0}^{\infty} \nu(k) = 1$$

which implies state 0 is recurrent.

(c) We use $\sum_{k=0}^{\infty} \nu(k) = 1$ to show

$$\begin{aligned}
\mu(n) &= \sum_{k\in\mathbb{N}} \mu(k)P(k, n) = \mu(0)\nu(n) + \mu(n+1) \\
&= \nu(n)\sum_{k=0}^{\infty} \nu(k) + [\mu(n) - \nu(n)] \\
&= \mu(n).
\end{aligned}$$

Therefore $\mu^\top P = \mu^\top$ and the measure defined by $\mu(n) = \nu(\{n, n+1, \cdots\}), n \in \mathbb{N}$ is stationary.